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CALCULATION OF VERTICAL DYNAMIC CHARACTERISTICS OF TALL BUILDINGS WITH VISCOUS DAMPING

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Abstract—The magnitude of the vertical component of earthquake ground motion is often about one-third of the horizontal component. Thus, it is necessary to calculate vertical dynamic characteristics of tall buildings and high-rise structures in design stage for certain cases. In analysing free vibrations of tall buildings and high-rise structures, it is possible to regard such structures as a cantilever bar with variable cross-section. In this paper, the differential equations of free longitudinal vibrations (in vertical direction) of bars with variably distributed mass and stiffness considering damping effect are established. The damping coefficient of a bar is assumed to be proportional to its mass, and the general solutions of mode shapes of damped distributed parameter systems are reduced to Bessel's equations by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass. An approach to determine the natural frequencies and mode shapes in vertical direction for tall buildings with variably distributed stiffness and variably distributed mass is proposed. The presented method is also applicable to the free longitudinal vibration analysis without considering damping effect (damping coefficient in vibration equations is equal to zero). A numerical example shows that the computed values of the fundamental longitudinal natural frequency and mode shape by the proposed method are close to the full scale measured data. It is shown through the numerical example that the selected expressions are suitable for describing the distributions of stiffness and mass of typical tall buildings. A comparison between undamped structural dynamic characteristics and damped natural frequencies, mode shapes is made in this paper. © 1998 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

All structures dissipate energy when they vibrate. Damping is a measure of the capacity of a system to dissipate the kinetic energy associated with induced vibration. So, damping is present to some degree in all structural systems. However, in general, the damping term in free vibration equations is omitted and the effect of damping on structural natural frequency and vibration mode shape is neglected in free vibration analysis. Although in the majority of engineering systems this effect is small and may be disregarded, there are cases where the effect reaches an appreciable magnitude and must be included in the analysis, for example, it is possible that the damping factor of a controlled structure is twenty times or more greater than that of corresponding uncontrolled structures in some cases (Soong, 1990). Therefore, there is a need to carry out further research on the evaluation of free vibration of structural systems considering damping effect.

In analysing free vibrations of tall buildings and high-rise structures, it is possible to regard such structures as a cantilever bar with variable cross-section. Wang (1978) investigated the free flexural vibration of a bar with variably distributed stiffness, but uniformly distributed mass. Li *et al.* (1994a, 1995) and Li (1995) studied the free flexural vibrations of tall buildings and high-rise structures which have variably distributed stiffness and mass. Li (1996) used several approximate computational methods to determine structural dynamic characteristics in vertical direction for tall buildings which are treated as one-step or multi-step bars. However, the damping terms in the equations of free vibration were omitted by these researchers. The computational method for analysis of free flexural vibrations of a bar with variably distributed stiffness, mass and damping was proposed by Li *et al.* (1997).

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But, free longitudinal vibration analysis of a cantilever bar with variably distributed damping, variably distributed stiffness and variably distributed mass by using analytical method has received little, if any, attention in the literature in the past. The exact solution of this problem has not previously been proposed in the literature. It is worth noting that it has been recognised that the magnitude of the vertical component of ground motion is often about one-third of the horizontal component. Wang (1978) reported that the vertical component of ground motions has a significant effect on earthquake induced responses of structures. Therefore, more work is thus required to determine the natural frequencies and mode shapes in vertical direction for tall buildings with variably distributed damping, variably distributed stiffness and variably distributed mass.

It is usually assumed that the mass of a tall building or a high-rise structure is proportional to its stiffness (e.g., Wang, 1978; Li *et al.*, 1994a, 1995) in free vibration analysis. This calculation procedure is reasonable for a part of high-rise structures, but it is not suitable for tall buildings and many high-rise structures, because the mass of floors is 80% or even more of the total mass of a tall building and the variation of mass at different floors is not significant, so, the mass distribution with height is almost constant for many cases, suggesting that the value of mass of a tall building is not necessarily proportional to its stiffness. This is confirmed by a series of shaking tests on buildings of various types in which the mass and stiffness of individual buildings have been measured and reported (Jeary and Sparks, 1977; Ellis and Jeary, 1980). In this paper, an approach to determine the natural frequencies and mode shapes in vertical direction for tall buildings with variably distributed damping, variably distributed stiffness and variably distributed mass, which are treated as bars with variable cross-section, is proposed.

Although many structures may be approximated by lumped mass system, in reality all structures are distributed mass systems having an infinite number of degrees of freedom. In this paper, free vibrations of damped distributed mass systems which represent tall buildings are presented and discussed.

EQUATIONS OF FREE LONGITUDINAL VIBRATIONS

The general differential equation for longitudinal (or axial) vibration of a bar with variable cross-section considering damping effect (Fig. 1) can be written as

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial y}{\partial x}\right) = \bar{m}_x \frac{\partial^2 y}{\partial t^2} + C_x \frac{\partial y}{\partial t} + p(x, t) \quad (1)$$

in which $y, p(x, t), K_x, C_x$ and $\bar{m}_x$ are the displacement in the longitudinal direction (vertical direction), the intensity of axial force, axial stiffness, viscous damping coefficient and mass per unit length, respectively, as section x.

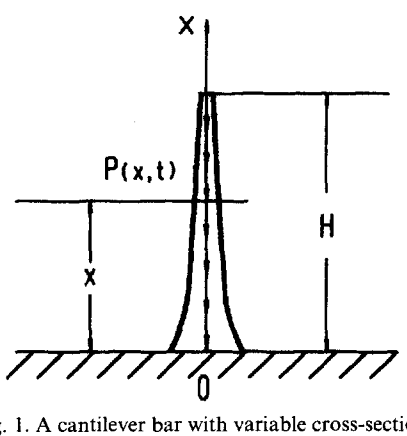

Fig. 1. A cantilever bar with variable cross-section.

If $p(x, t) = 0$, then eqn (1) becomes the equation of free longitudinal vibration considering damping effect as follows

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial y}{\partial x}\right) - \bar{m}_x \frac{\partial^2 y}{\partial t^2} - C_x \frac{\partial y}{\partial t} = 0 \tag{2}$$

If set

$$y(x, t) = X(x)\exp(\lambda t) \tag{3}$$

$$C_x = C_0 \bar{m}_x \tag{4}$$

Then, the equation of mode shape function $X(x)$ is given by

$$K_x \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{\mathrm{d}K_x}{\mathrm{d}x}\frac{\mathrm{d}X}{\mathrm{d}x} + \bar{m}_x \omega^2 X = 0 \tag{5}$$

where

$$\omega^2 = -\lambda(C_0 + \lambda) \tag{6}$$

It is difficult to find the exact solutions of eqn (5) for general cases, because the structural parameters in the equation vary with the coordinate x. It is obvious that the exact solutions are dependent on the distributions of damping, mass and stiffness. Thus, the exact solution of eqn (5) may be obtained by means of reasonable selections for mass and stiffness distributions. As suggested by Tuma and Cheng (1983), the functions which can be used to approximate the variation of mass and stiffness are algebraic polynomials, exponential functions, trigonomeric series, or their combinations. In this paper, two important cases are considered and discussed as follows.

Case A: Expressions of mass and axial stiffness are power functions

$$K_x = \alpha(1 + \beta x)^{\gamma} \tag{7}$$

$$\bar{m}_x = a(1 + \beta x)^{c} \tag{8}$$

in which α, β and γ are constants which can be determined by use of the real values of the axial stiffness at $x = 0$, $H/2$ and H as follows

$$\alpha = K_0$$
$$\beta = \frac{1}{H}\left[\left(\frac{K_H}{K_0}\right)^{1/\gamma} - 1\right]$$
$$\gamma = \frac{\ln K_{H/2} - \ln K_0}{\ln\left(1 + \dfrac{\beta H}{2}\right)} \tag{9}$$

Similarly, a, β, and c can be found as

$$a = m_0$$
$$\beta = \frac{1}{H}\left[\left(\frac{m_H}{m_0}\right)^{1/c} - 1\right]$$

$$c = \frac{\ln(m_{H/2}) - \ln(m_0)}{\ln\left(1 + \frac{\beta H}{2}\right)} \tag{10}$$

where m_0, K_0, $m_{H/2}$, $K_{H/2}$, m_H, K_H are the mass intensity and the axial stiffness, respectively, at $x = 0$, $H/2$, and H. H is the height of the structure considered (Fig. 1). It is worth noting that the value of β determined from eqn (9) may not be equal to that of β calculated from eqn (10), one can take their average as the representative value of β.

Case B: *Expressions of mass and axial stiffness are exponential functions*

$$K_x = \alpha\, e^{-\beta(x/H)} \tag{11}$$

$$m_x = a\, e^{-b(x/H)} \tag{12}$$

The parameters α, β, a, b can be determined by

$$\alpha = K_0, \quad \beta = \ln(K_0) - \ln(K_H) \tag{13}$$

$$a = m_0, \quad b = \ln(m_0) - \ln(m_H) \tag{14}$$

SOLUTIONS OF THE DIFFERENTIAL EQUATIONS

Case A

Substituting eqns (7) and (8) into eqn (5) gives

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{\gamma\beta}{1+\beta x}\frac{\mathrm{d}X}{\mathrm{d}x} + \frac{a\omega^2}{\alpha}(1+\beta x)^{c-\gamma} X = 0 \tag{15}$$

Setting

$$X = \left[\frac{(c-\gamma+2)}{2n}\xi\right]^{v} \Psi$$

$$\xi = \frac{2n}{c-\gamma+2}(1+\beta x)^{\frac{c-\gamma+2}{2}}$$

$$v = \frac{1-\gamma}{c-\gamma+2}$$

$$n^2 = \frac{a\omega^2}{\alpha\beta^2} \tag{16}$$

and substituting eqn (16) into eqn (15) gives

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} + \left(1 - \frac{v^2}{\xi^2}\right)\Psi = 0 \tag{17}$$

The above equation is a Bessel's equation of the v-th order. The vibration mode function for a non-integer, v can be expressed as

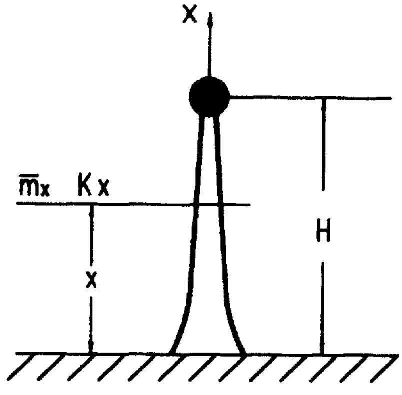

Fig. 2. A cantilever bar with a lumped mass.

$$X = \left(\frac{c-\gamma+2}{2n}\right)^{v} [c_1 J_v(\xi) + c_2 J_{-v}(\xi)] \tag{18}$$

where $J_v(\xi)$ is the Bessel function of the first kind of order v.

The boundary conditions of the structure shown in Fig. 2 are as follows

$$\begin{aligned} x &= 0, \quad X(0) = 0 \\ x &= H, \quad K_H \left(\frac{\mathrm{d}X}{\mathrm{d}x}\right)_{x=H} = m\omega^2 X(H) \end{aligned} \tag{19}$$

Substituting the above boundary conditions eqn (19) into eqn (18) gives the following frequency equation

$$J_{-v}(N)\left[n\theta B J_{v-1}(N\theta) + \frac{n^2\alpha^2\beta^2 m}{a} J_v(N\theta)\right] = -J_v(N)\left[n\theta B J_{-(v-1)}(N\theta) - \frac{n^2\alpha^2\beta^2 m}{a} J_{-v}(N\theta)\right] \tag{20}$$

in which m is the lumped mass attached to the top of a cantilever bar with variable cross-section (Fig. 2).

If there is no lumped mass at the top of the bar, that is, $m = 0$, then the frequency equation for this case is given by

$$J_{-v}(N)J_{v-1}(N\theta) = -J_v(N)J_{-(v-1)}(N\theta) \tag{21}$$

where

$$\begin{aligned} N &= \frac{2n}{c-\gamma+2} \\ \theta &= (1+\beta H)^{\frac{c-\gamma+2}{2}} \\ B &= (1+\beta H)^{-\frac{1+\gamma}{2}} \end{aligned} \tag{22}$$

When v is an integer, the longitudinal vibration mode function becomes

$$X = \left(\frac{c-\gamma+2}{2n}\right)^v [c_1 J_v(\xi) + c_2 Y_v(\xi)] \tag{23}$$

The frequency equation takes the form

$$Y_v(\lambda)\left[n\theta B J_{v-1}(\lambda\theta) + \frac{n^2\alpha^2\beta^2 m}{a} J_v(\lambda\theta)\right] = J_v(\lambda)\left[n\theta B Y_{v-1}(\lambda\theta) - \frac{n^2\alpha^2\beta^2 m}{a} Y_v(\lambda\theta)\right] \tag{24}$$

or

$$Y_v(\lambda) J_{v-1}(\lambda\theta) = J_v(\lambda) V_{v-1}(\lambda\theta) \tag{25}$$

for $m = 0$

If $\gamma = c+2$, then eqn (15) is reduced to an Euler's equation as follows

$$(1+\beta x)^2 \frac{d^2 X}{dx^2} + \gamma\beta(1+\beta x)\frac{dX}{dx} + \frac{a\omega^2}{\alpha} X = 0 \tag{26}$$

The general solution of eqn (26) can be written as

$$X = (1+\beta x)^{\frac{1-\gamma}{2}}\{c_1 \cos[\sqrt{D}\ln(1+\beta x)] + c_2 \sin[\sqrt{D}\ln(1+\beta x)]\} \tag{27}$$

Using eqn (18) and the boundary conditions, gives the mode shape function as follows

$$X = c_2(1+\beta x)^{\frac{1-\gamma}{2}} \sin[\sqrt{D}\ln(1+\beta x)] \tag{28}$$

The frequency equation is

$$2\sqrt{D}\, ctn[\sqrt{D}\ln(1+\beta H)] = \gamma - 1 - \frac{2\beta m n^2}{a d^{\gamma-1}} \tag{29}$$

or

$$2\sqrt{D}\, ctn[\sqrt{D}\ln(1+\beta H)] = \gamma - 1 \tag{30}$$

for $m = 0$ in which

$$D = n^2 - \frac{(1-\gamma)^2}{4} > 0 \tag{31}$$

Because the case corresponding to $D < 0$ is meaningless, it is not considered here. The solution presented above is called the general solution for case A. The special cases can be found from the general solution as follows.

(1) When $\gamma = c$, $v = [(1-\gamma)/2]$, this case is corresponding to a bar that the mass of it is proportional to its longitudinal stiffness. In general, solid bars and some high-rise structures belong to this case. It is called the special case 1 here.
(2) When $\gamma \neq 0$, $c = 0$, this case represents a bar with variably distributed stiffness and uniformly distributed mass. The corresponding solution can be found from the general solution. Some tall buildings can be considered as this case called the special case 2 here.

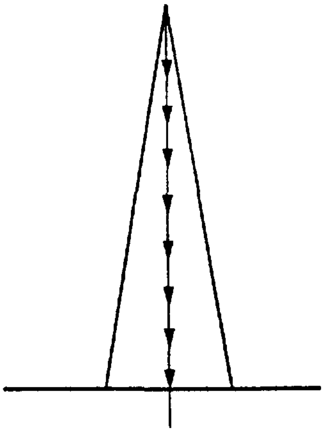

Fig. 3. A cuneiform bar.

(3) When $\gamma = 0, c \neq 0$, the general solution becomes that of a bar with uniformly distributed stiffness and variably distributed mass. This case is called the special case 3.
(4) When $\gamma = 0$ and $c = 0$, the general solution represents that of a uniform bar.
(5) When $\beta = (1/H)$, the general solution becomes the solution of a wedged bar with variably distributed stiffness and variably distributed mass (Fig. 3).

Case B

Substituting eqns (11) and (12) into eqn (5) gives

$$\frac{d^2X}{dx^2} - \frac{\beta}{H}\frac{dX}{dx} + \frac{a}{\alpha}\omega^2 e^{\frac{(\beta-b)x}{H}} X = 0 \tag{32}$$

Setting

$$\begin{aligned} X &= \xi^v Z \\ \xi &= e^{\frac{(\beta-b)x}{2H}} \\ v &= \frac{\beta}{\beta-b} \\ s^2 &= \frac{4a\omega^2 H^2}{\alpha(\beta-b)^2} \end{aligned} \tag{33}$$

and substituting eqn (33) into eqn (32) lead to

$$\frac{d^2Z}{d\xi^2} + \frac{1}{\xi}\frac{dZ}{d\xi} + \left(s^2 - \frac{v^2}{\xi^2}\right)Z = 0 \tag{34}$$

Equation (34) is a Bessel's equation of the v-th order. The vibration mode function for a non-integer v can be expressed as

$$X = \xi^v[c_1 J_v(s\xi) + c_2 J_{-v}(s\xi)] \tag{35}$$

where $J_v(s\xi)$ is the Bessel function of the first kind of order v.

Using eqn (18) and the boundary conditions obtains the following frequency equation

$$J_{-v}(s)\left[\frac{(\beta-b)sm}{2aH}J_v(sA)+A\,e^{-\beta}J_{v-1}(sA)\right]=J_v(s)\left[\frac{(\beta-b)sm}{2aH}J_{-v}(sA)-A\,e^{-\beta}J_{-(v-1)}(sA)\right] \tag{36}$$

in which

$$A=e^{\frac{\beta-b}{2}} \tag{37}$$

When $m = 0$, the frequency equation becomes

$$J_{-v}(s)J_{v-1}(sA) = -J_v(s)J_{-(v-1)}(sA) \tag{38}$$

If v is an integer, the vibration mode function can be expressed as

$$X = \xi^v[c_1 J_v(s\xi)+c_2 Y_v(s\xi)] \tag{39}$$

where $Y_v(s\xi)$ is the Bessel function of the second kind of order v.

The longitudinal frequency equation is given by

$$Y_v(s)\left[\frac{(\beta-b)sm}{2aH}J_v(sA)+A\,e^{-\beta}J_{v-1}(sA)\right]=J_v(s)\left[\frac{(\beta-b)sm}{2aH}Y_v(sA)+A\,e^{-\beta}Y_{v-1}(sA)\right] \tag{40}$$

When $m = 0$, the frequency equation becomes

$$Y_v(s)J_{v-1}(sA) = J_v(s)Y_{v-1}(sA) \tag{41}$$

The solution obtained from the above equation is called the general solution for case B.

The following special cases can be found from the general solution.

(1) When $\beta = b$, the general solution becomes that of the special case 1 defined above. Equation (34) is reduced to a differential equation with constant coefficients in this case.
(2) When $\beta \neq 0$, $c = 0$, the solution of the special case 2 can be obtained from the general solution, in this case $v = 1$.
(3) When $\beta = 0$, $c \neq 0$, the solution of the special case 3 can be obtained from the general solution.
(4) When $\beta = 0$ and $c = 0$, the general solution becomes that of a uniform bar.

NUMERICAL EXAMPLE

A tall building (27 stories) which is located in Guangzhou is a shear-wall structure with variable cross-section. Based on the full-scale measurement of free vibration of this building (Li *et al.*, 1994a), this building can be treated as a cantilever bar (Fig. 1) in free vibration analysis. The procedure for determining the dynamics characteristics of this tall building with viscous damping ($C_0 = 0.2$ = critical damping ratio = $\xi = 0.03$) in vertical direction is as follows:

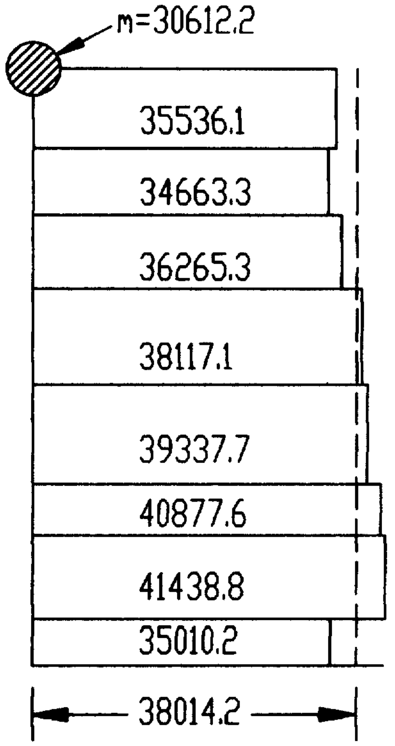

Fig. 4. Mass distribution of the tall building.

1. *Determination of the mass per unit length* (*Fig.* 4)

The axial stiffness and mass per unit length (Fig. 4) of the building vary monotonically downward with height. For simplicity, the building is treated as a cantilever bar with variable cross-section as shown in Fig. 1. Because the variation of the mass per unit length and the lumped mass attached at the top of the building are comparative small, it is reasonable to assume that the mass is uniformly distributed along the height of the building (Fig. 4).

The mass per unit length, $\bar{m}$, is found as: $\bar{m} = 38{,}014.2$ kg/M.

2. *Evaluation of the axial stiffness*, K_x, (Fig. 5)

For this example, the distributions of mass and axial stiffness per unit length along the building height are described as power functions [eqn (7) and eqn (8)], which are given as

$$K_x = \alpha(1+\beta x)^{\gamma} \tag{42}$$

$$\bar{m}_x = a(1+\beta x)^{c} \tag{43}$$

Because the mass is considered as uniformly distributed, it is suggested that $c = 0$ in eqn (43). The stiffness distribution is taken as

$$K_x = \alpha(1+\beta x)^{2} \tag{44}$$

According to the boundary conditions of this building:

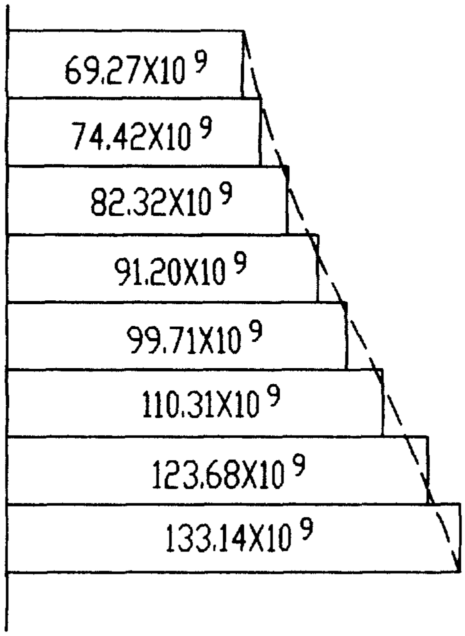

Fig. 5. Stiffness distribution of the tall building.

$$\text{at } x = 0, \quad EF_0 = 133.14 \times 10^9\, KN$$
$$x = H, \quad EF_H = 69.27 \times 10^9\, KN$$

The parameters, α, β, are determined as

$$\alpha = EF_0 = 133.14 \times 10^9\, KN$$
$$\beta = -3.667 \times 10^{-3}$$

The evaluated distribution of stiffness is shown in Fig. 5 (dotted line).

3. *Evaluation of the fundamental natural frequency*

The frequency equation (30) becomes

$$2\sqrt{D}\, ctn[\sqrt{D} \ln(1+\beta H)] = 1 \tag{45}$$

Solving the above equation gives

$$\sqrt{D} = 5.0855$$

The corresponding axial fundamental frequency is found from eqn (31) as follows

$$\omega_1 = 35.06 \text{ rad/s}$$

$f_1 = 5.580$ Hz (the first undamped natural frequency).

Using eqn (6) gives

$$\lambda_{1,2} = -0.05 \pm 35.03i$$

The first damped natural frequency is

Table 1. Longitudinal fundamental mode shape of the 27-storey building

Storey level	1	2	5	8	11	14	17	20	24
x/H	0	0.0704	0.2007	0.3230	0.4454	0.5678	0.6976	0.8125	1
$Y_1(x/H)$ measured	0	0.100	0.257	0.417	0.560	0.710	0.837	0.929	1
$Y_1(x/H)$ calculated ($\xi = 0.03$)	0	0.1022	0.2687	0.4269	0.5687	0.7147	0.8475	0.9375	1
$Y_1(x/H)$ calculated ($\xi = 0$)	0	0.1022	0.2687	0.4269	0.5687	0.7147	0.8475	0.9375	1

$$\omega_1 = 35.03\,\text{rad/s}$$

$f_1 = 5.575$ Hz (the first damped natural frequency).

It can be seen that the difference between the undamped frequency and damped frequency is very small.

The longitudinal fundamental frequency obtained by full-scale measurement (Li *et al.*, 1994a) is 5.47 Hz. It is clear that the computed values in terms of the proposed procedure are in good agreement with the measured data.

If the lumped mass attached to the top of the building ($M = 30{,}612.2$ kg) is considered, then, the axial frequency eqn (29) must be used. The calculated axial fundamental undamped frequency is found as 5.570 Hz, and the fundamental damped frequency is 5.565 Hz.

4. *Calculation of the longitudinal vibration mode shape*

After computing the first natural frequency f_1, the first mode shape, $X_1(x)$, can be determined from eqn (22). The calculated results are listed in Table 1. Using the aforementioned procedure, the higher natural frequencies and corresponding mode shapes can also be determined.

It can be seen from Table 1 that the calculated fundamental mode shapes show good agreement with the measured mode shape. The computed model shape without including the damping term ($\xi = 0$) is the same as that calculated considering the damping effect ($\xi = 0.03$). Therefore, it can be concluded that if the distribution of damping coefficient is assumed to be proportional to that of mass ($C_x = C_0\bar{m}_x$), there is no effect of damping on the fundamental mode shape.

CONCLUSION

An approach to determine the longitudinal natural frequencies and mode shapes of tall buildings with viscous damping, variably distributed stiffness and variably distributed mass, which are treated as cantilever bars with variable cross-section, is proposed. The proposed formulae for determining free longitudinal vibrations of tall buildings are simple and convenient for engineering applications. The numerical example showed that the calculated fundamental longitudinal natural frequency and mode shape of a 27-storey tall building are very close to the full scale measured data, suggesting that the calculation method proposed in this paper are applicable to engineering application and practice. The computed results also showed that if the distribution of damping coefficient is assumed to be proportional to that of mass ($C_x = C_0\bar{m}_x$), the difference between the first damped frequency and the first undamped natural frequency is very small and there is no damping effect on the longitudinal fundamental mode shape. It has been shown through the numerical example that the selected expressions are suitable for describing the distribution of stiffness and mass of typical tall buildings. Therefore, the proposed method is applicable to the free longitudinal vibration analysis of tall buildings with or without considering damping effect.

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REFERENCES

Jeary, A. P. and Sparks, P. R. (1977) Some observations on the dynamic sway characteristics of concrete structures. *Proceedings of Symposium on Vibrations in Concrete Structures*. New Orleans.

Ellis, B. R. and Jeary, A. P. (1980) A review of recent work on the dynamic behaviour of tall buildings at various amplitudes. *Proceedings of the 7th World Conference on Earthquake Engineering*. Istanbul.

Li, Q. S. (1995) Calculation of free vibration of high-rise structures. *Asian Journal of Structural Engineering* **1**(1), 17–25.

Li, Q. S. (1996) Free axial vibrations of tall buildings and high-rise structures. *International Journal of Computers and Structures*, submitted.

Li, Q. S., Cao, H. and Li, G. (1994a) Analysis of free vibrations of tall buildings. *ASCE, Journal of Engineering Mechanics* **120**(9), 1861–1876.

Li, Q. S., Cao, H. and Li, G. (1994b) Stability calculation of bars with multi-step varying cross section. *International Journal of Computers and Structures* **53**(5), 1085–1089.

Li, Q. S., Cao, H. and Li, G. (1995) Stability analysis of bars with varying cross section. *International Journal of Solids and Structures* **32**(21), 3217–3228.

Li, Q. S., Fang, J. Q., Jeary, A. P. and Wong, C. K. (1997) Computation of free flexural vibrations of tall buildings and high-rise structures considering damping effect. *Journal of Sound and Vibrations*, submitted.

Soong, T. T. (1990) Active structural control, theory and practice. *Longman Scientific and Technical*, pp. 18–43.

Tuma, J. J. and Cheng, F. Y. (1983) Dynamic structural analysis. McGraw-Hill Company.

Wang, G. Y. (1978) Vibration of building and structures. *Science and Technology Press*, pp. 168–178. Beijing.

APPENDIX

The following symbols are used in this paper:

y	displacement in the longitudinal direction
$p(x, t)$	intensity of axial force
K_x	axial stiffness
C_x	viscous damping coefficient
$X(x)$	mode shape function
ω	circular natural frequency
$\bar{m}_x$	mass per unit length
H	height of the structure.